1. Exercises from Sections 2.4-2.5

PROBLEM 1. *(Folland 2.4.5) Suppose S is a convex, connected open set and $f : S \to \mathbb{R}$ is differentiable with $\partial_1 f = 0$ for all $x \in S$, then $f(a) = f(b)$ for all $a, b \in S$ such that $a_j = b_j$ for all $j \neq 1$.*

PROOF. We prove by contrapositive.

(1) Draw picture
(2) Suppose that f is not independent of x_1, so that there exist points $a, b \in S$ with $a \neq b$ and $f(a) \neq f(b)$, but $a_j = b_j$ for every $j \neq 1$.
(3) By convexity and connectedness of S, there is a line segment L joining a to b completely contained in S.
(4) Applying the mean value theorem, there exists a point $c \in L$ such that

$$f(b) - f(a) = \nabla f(c) \cdot (b - a)$$

(5) $b - a = (b_1 - a_1)\hat{x}_1$, since $a_j = b_j$ for all $j \neq 1$
(6) By differentiability of f,

$$\nabla f(c) \cdot (b - a) = \left.\frac{\partial f}{\partial x_1}\right|_{x=c} (b_1 - a_1)$$

(7) Now we just estimate:

$$\left| \left.\frac{\partial f}{\partial x_1}\right|_{x=c} \right| = \frac{|f(b) - f(a)|}{|b_1 - a_1|} > 0$$

(8) So there exists a point $c \in S$ at which $\partial_1 f \neq 0$. This completes the proof.

□

Explain the counter example when S is not convex or connected by drawing some pictures.